Dynamic Optimization and Mathematical Economics

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Dynamic Optimization and Mathematical Economics

Edited by Pan-Tai Liu

University of Rhode Island Kingston, Rhode Island

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Preface

As an outgrowth of the advancement in modern control theory during the past 20 years, dynamic modeling and analysis of economic systems has become an important subject in the study of economic theory. Recent developments in dynamic utility, economic planning, and profit optimization, for example, have been greatly influenced by results in optimal control, stabilization, estimation, optimization under conflicts, multicriteria optimization, control of large-scale systems, etc.

The great success that has been achieved so far in utilizing modern control theory in economic systems should be attributed to the effort of control theorists as well as economists. Collaboration between the two groups of researchers has proven to be most successful in many instances; nevertheless, the gap between them has existed for some time. Whereas a control theorist frequently sets up a mathematically feasible model to obtain results that permit economic interpretations, an economist is concerned more with the fidelity of the model in representing a realworld problem, and results that are obtained (through possibly less mathematical analysis) are due largely to economic insight.

The papers appearing in this volume are divided into three parts. In Part I there are five papers on the application of control theory to economic planning. Part II contains five papers on exploration, exploitation, and pricing of extractive natural resources. Finally, in Part III, some recent advances in large-scale systems and decentralized control appear.

These papers are contributions from control theorists and economists. Each paper presents its own perspective of future developments in dynamic economic theory. It is hoped that this collection will help stimulate interaction between the control-theoretic and the economically oriented approaches and that the volume will be of interest to researchers in applied mathematics, economics, management sciences, etc. It is also hoped that this collection will provide a periscopic view of some recent progress in mathematical economics. In editing this volume we have been assisted by Dr. Tamer Başar, Professor Jon Sutinen, and Professor Henry Wan. Most of the papers in this volume were presented at the Third Kingston Conference on Differential Games and Control Theory, held in June 1978; the theme of the conference was dynamic optimization and mathematical economics. Financial support from the Office of Naval Research and the University of Rhode Island in organizing the conference are gratefully acknowledged. Finally, we would like to thank Professor George Leitmann and Professor Angelo Miele for making possible the publication of this volume in the present series.

Kingston, Rhode Island

Pan-Tai Liu

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Part I

Control Theory in Economic Systems

The dynamic feature of economic systems has become increasingly important as the economic systems of modern society become more complex. The concept of state variables, which summarize the past history of a system, is essential for describing problems in economics, while control variables play the role of intertemporal decision making. Originally developed by engineers and mathematicians, control theory is now widely used in economics.

The importance of control theory in economics can be demonstrated by the extent of the literature that has been published, especially during the last ten years. Recent advances in economic planning, growth theory, and other aspects of the theory of firms have resulted from applications of modern control techniques.

The objectives in control of an economic system include optimization, stabilization, and a combination of the two. Optimization refers to maximization of one or more payoffs, such as profit flow and utility levels of capital assets at the end of the planned horizon. Stabilization means steering the state of the system (demand, price, laborers, or capital) toward a target level by proper choice of the control (investment, production, or fiscal policy).

The evolution of an economic system in general can be described by accumulation equations, which are equivalent to state equations in control theory. The objective functional is often the integral of an instantaneous utility function over a planned horizon. The problem is then analyzed by standard techniques in optimal control, e.g., the maximum principle and dynamic programming.

While quantitative results that bear some economic significance are desirable, they are not always obtainable. In many cases, qualitative interpretation of necessary conditions (for optimality) can be obtained with a moderate amount of mathematical analysis. A fundamental example is given by interpreting the adjoint variable in the maximum principle as the shadow price, which weighs future accumulation in the system against present utility. In general, a certain amount of economic insight is needed to "penetrate" a set of mathematical conditions and formulate some underlying economic theory.

A special feature of the control problem in economics is that the planned horizon may be infinite. In such a case, the transversality condition, which is part of the necessary condition for optimality, has to be formulated and utilized in the limiting sense as time goes to infinity. Steady-state solutions stand for some kind of dynamic equilibrium in the long run. They can be obtained (with the aid of transversality conditions) by setting all the derivatives with respect to time equal to zero. This is discussed in the papers by Chang and Leland.

Chang's paper illustrates that the standard optimal control theory can be applied to diverse economic problems. Three examples are given: (1) a comprehensive energy-planning problem for the government, (2) the optimal investment problem for a computer-using firm, in which the investment in research and development takes the form of developing programs, subroutines, etc., and (3) the optimal saving problem for the consumers, in which the impact of changing the initial capital value is analyzed in detail.

Leland's paper applies the conventional control model to a novel problem in the theory of firms. Suppose that real-life firms maximize some objective functions other than profit. Then for a large class of cases, the optimal decisions of the firm approach the optimal policies under profit maximization in the long run. This is in contrast to results from static models, which indicate that the decisions of a firm with alternative objectives differ from the decisions based on profit maximization.

The theory of finance deals with many aspects of a firm's performance as a corporation. One of the most important problems is the optimal investment and consumption plan over time. This includes (1) the amounts of investment and consumption at any time and (2) the allocation of investment among different types of assets.

Brock's paper contains a survey of his works and related works on asset pricing. Starting with an N-process stochastic growth model, described by stochastic difference equations, he characterizes optimal paths generated by optimum policies by means of the principle of optimality. If all consumers are assumed to be identical, their optimal consumptioninvestment decisions imply that production must be carried out in a certain fashion for a multigood model, buffeted by random shocks from nature. The profit and the stock price of the firms carrying out the production processes can then be derived. The limiting distribution of capital assets corresponds to the limiting distribution of stock prices.

For the one-consumer case, the N-process growth model is then

Introduction

converted into an asset-pricing model by introducing competitive rental markets for the capital goods and a market for pure rents generated by each individual firm. A general economic theory, based on the concept of a rational expectation equilibrium, is established and a unique assetpricing function is indicated to exist.

In the modeling of an economic system, it is frequently necessary to consider uncertainties. One way to do so is to take the probabilistic approach of assuming that some parameters in the system are random variables or that the system is subject to some disturbances that can be described as stochastic processes. We then have a stochastic control problem in which the stochastic maximum principle or stochastic dynamic programming can be applied. For example, the fluctuation of the stock market can be described as a Brownian motion, and the accumulation equation for the capital of a firm then becomes an Ito's equation. Mathematical theory in control of dynamic systems described by Ito's equations has been fairly well established. The application of such a theory in economics is currently an area of active research.

Another way to describe uncertainties is to assume that they are bounded in some way but are otherwise unknown. The optimization problem is then formulated as one of finding the feedback control from within a certain class and minimizing the maximum values over all uncertain quantities. This worst-case, nonstochastic approach is often more natural and realistic, though the actual computation of a minimax control is generally complicated. The same principle can be applied to stabilization against uncertain disturbances by minimizing the maximum possible values of a Lyapunov function. The paper by Leitmann and Wan follows this approach in discussing the stabilization of a macroeconomic system that contains some unknown characteristics. They show that under some conditions, global, asymptotic stability can be guaranteed, uniformly over a class of bounded disturbances. If these conditions are absent, due either to observational error or to delay, or due to limitation of control instruments in their magnitude or scope, the performance of the economy can still be improved by adopting certain policies.

In large firms, pricing and resource allocation are among the most important aspects of decisionmaking. Shubik and Sobel define discrete time sequential games as multiperson Markov decision processes. They use such game models to describe dynamic oligopolistic market situations and other competitive resource allocation problems. In addition to addressing themselves to some issues that arise in such models, they discuss some principal sufficient conditions for optimality satisfied by various dynamic oligopoly models. 1

Asset Pricing in an Economy with Production: A "Selective" Survey of Recent Work on Asset-Pricing Models

WILLIAM A. BROCK

1. Introduction

This paper surveys an intertemporal general equilibrium theory of capital asset pricing. It is an attempt to put together ideas from the literatures on modern finance, stochastic growth models, and general equilibrium theory. In this way we shall obtain a theory capable of addressing general equilibrium questions such as the following: What is the impact of an increase in the corporate income tax on the relative prices of risky stocks? What is the impact of an increase in progressivity of the personal income tax on the relative price structure of risky assets? This paper discusses only recent literature that is closely related to my own work. Hence it should be read with this disclaimer in mind. Furthermore, because of space limitations, theorems and proofs will be loosely stated.

The theory presented here derives part of its inspiration from Merton (Ref. 1). However, Merton's intertemporal capital asset-pricing model (ICAPM) is not a general equilibrium theory in the sense of Arrow-Debreu. That is, the *technological* sources of uncertainty are not related to the equilibrium prices of the risky assets in Merton (Ref. 1). We do that here and preserve the empirical tractability of Merton's formulation.

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One strand of the work surveyed here in the one-consumer case modifies the stochastic growth models of Brock and Mirman (Refs. 2, 3), Radner (Ref. 4), and Jeanjean (Ref. 5) in order to put a nontrivial investment decision into the asset-pricing model of Lucas (Ref. 6). Brock (Ref. 7) does this in a way that preserves the empirical tractability of the Merton formulation and at the same time determines the risk prices derived by Ross (Ref. 8) in his arbitrage theory of capital asset pricing. Ross's price of systematic risk k at date t, denoted by λ_{kt} and induced by the source of systematic risk δ_{kt} , is determined by the covariance of the marginal utility of consumption with λ_{kt} . In this way we discover assumptions about tastes and technology needed to generate Ross's λ_{kt} and show exactly how they are determined by the interaction of sources of production uncertainty and the demand for risky assets.

Prescott and Mehra (Ref. 9) insert an investment decision into Lucas's model (Ref. 6) and develop a dynamic programming recursive equilibrium framework that covers many recent models of asset pricing such as Brock (Ref. 7) and Lucas (Ref. 6).

Another line of work is developed in continuous time by Cox, Ingersoll, and Ross (Ref. 10), and others. This work seeks to combine intertemporal general equilibrium analysis with the pricing theory of derivative claims that culminated in the work of Black and Scholes (Ref. 11).

This paper proceeds as follows. Section 1 contains the introduction. Section 2 presents an N-process stochastic growth model, which forms the basis for the quantity side of the asset-pricing model developed in Section 3 for the one-consumer case. It will lead to useful insights for the multiconsumer case.

Section 2 describes optimum paths generated in the N-process model by time-independent continuous optimum policy functions a la Bellman. Sufficient conditions on tastes and technology are developed that allow the derivation of a functional equation that determines the state valuation function, with the use of methods adapted from a paper by Koopmans (Ref. 12). We also discuss the problem of finding sufficient conditions on tastes and technology such that for any initial state the optimum stochastic process of investment converges in distribution to a limit distribution independent of the initial state.

For the one-consumer case Section 3 converts the growth model of Section 2 into an asset-pricing model by introducing competitive rental markets for the capital goods and by introducing a market for claims to the pure rents generated by the *i*th firm, i = 1, 2, ..., N. Each of the N processes is identified with one "firm." Firms pay out rentals to consumers. The residual is pure rent. Paper claims to the pure rent generated by each firm *i* and a market for these claims are introduced along the line of Lucas (Ref. 6), i.e., both sides of the economy possess subjective distributions on pure rents, capital rental rates, and share prices. Both sides draw up demand and supply schedules conditioned on their subjective distributions. Market clearing introduces an objective distribution on pure rents, capital rental rates, and share prices. A rational expectations equilibrium, abbreviated R.E.E., is defined by the requirement that the objective distribution equal the subjective distribution at each date.

Section 3 shows (for the one-consumer case), with the use of recent results of Benveniste and Scheinkman (Ref. 13), that the quantity side of an R.E.E. is identical to the quantity side of the N-process growth model developed in Section 2. The key idea used is the Benveniste–Scheinkman result that for utility functions that are additively separable over time, the standard transversality condition at infinity is necessary as well as sufficient for an infinite-horizon concave programming problem. This theory is used to characterize optimal plans by each consumer.

The financial side of the economy is now easy to develop. A unique asset-pricing function for stock i of the form $P_i(y)$ is indicated to exist by use of a contraction mapping argument along the line of Lucas (Ref. 6), where y describes the state of the economy (which may include its history).

Section 4 reviews some other asset-pricing models. The first model, developed by Becker (Ref. 14), is a deterministic version of the one-consumer model developed in Section 3. He shows that if each consumer's utility function is a discounted sum of period utilities with timeindependent discount factor and time-independent period utility function, then the consumer that discounts the future the least eventually ends up with all the capital stock.

The next paper, by Magill (Ref. 15), applies the recent work of Bismut (Ref. 16) on continuous-time stochastic optimal control to economic dynamics. Magill's paper uses Bismut's work to characterize rational expectations equilibria in an economic world driven by Ito processes.

An important class of continuous-time stochastic asset-pricing models is reviewed in Section 5. These models, built by Cox, Ingersoll, and Ross (Ref. 10; hereafter CIR) are general equilibrium versions of Merton's intertemporal capital asset-pricing model (Ref. 1). CIR also add a class of derivative claims and develop generalized Black-Scholes (Ref. 11) partial differential equations for the prices of these claims.

Finally, Section 6 contains a summary.

2. The Optimal Growth Model

Since the optimal growth model is studied in detail in Brock (Ref. 7), we shall be brief where possible.

The model is given by the following:

Maximize

$$E_{1}\sum_{t=1}^{\infty}\beta^{t-1}u(c_{t}), \qquad (1)$$

subject to

$$c_{t+1} + x_{t+1} - x_t = \sum_{i=1}^{N} [g_i(x_{it}, r_t) - \delta_i x_{it}], \qquad (2)$$

$$x_t = \sum_{i=1}^{N} x_{it}, \quad x_{it} \ge 0, \quad i = 1, 2, \dots, N, t = 1, 2, \dots,$$
 (3)

$$c_t \ge 0, \qquad t = 1, 2, \dots,$$
 (4)

$$x_0, x_{i0}$$
 $(i = 1, 2, \dots, N), r_0$ historically given, (5)

where E_1 , β , u, c_i , x_i , g_i , x_{it} , r_t , δ_i denote, respectively, mathematical expectation conditioned at time 1, discount factor on future utility, utility function of consumption, consumption at date t, capital stock at date t, production function of process i, capital allocated to process i at date t, random shock which is common to all processes i, and depreciation rate for capital installed in process i.

The space of $\{c_t\}_{t=1}^{\infty}$, $\{x_t\}_{t=1}^{\infty}$ over which the maximum is being taken in (1) needs to be specified. Obviously decisions at date t should be based only on information at date t. In order to make the choice space precise some formalism is needed. We borrow (copy) from Brock and Majumdar (Ref. 17) at this point.

The environment will be represented by a sequence $\{r_t\}_{t=1}^{\infty}$ of real vector-valued random variables which will be assumed to be independently and identically distributed. The common distribution of r_t is given by a measure $\mu: \mathcal{B}(\mathbb{R}^m) \to [0, 1]$, where $\mathcal{B}(\mathbb{R}^m)$ is the Borel σ -field of \mathbb{R}^m . In view of a well-known one-to-one correspondence [see, e.g., Loeve (Ref. 18, pp. 230-231)], we can adequately represent the environment as a measure space $(\Omega, \mathcal{F}, \nu)$, where Ω is the set of all sequences of real *m*-vectors, \mathcal{F} is the σ -field generated by cylinder sets of the form $\prod_{t=1}^{\infty} A_t$, where

$$A_t \in \mathcal{B}(\mathbb{R}^m), \quad t = 1, 2, \dots,$$

 $A_t = \mathbb{R}^m$

and

for all but a finite number of values of t. Also ν (the stochastic law of the environment) is simply the product probability induced by μ (given the assumption of independence).

The random variables r_t may be viewed as the *t*th coordinate function on Ω , i.e., for any $\omega = \{\omega_t\}_{t=1}^{\infty} \in \Omega$, $r_t(\omega)$ is defined by

$$r_t(\omega) = \omega_t$$

We shall refer to ω as a possible state of the environment (or an environment sequence) and to ω_t as the environment at date *t*. In what follows, \mathscr{F}_t is the σ -field guaranteed by partial histories up to period *t* (i.e., the smallest σ -field generated by cylinder sets of the form $\prod_{\tau=1}^{\infty} A_{\tau}$ where A_{τ} is in $\mathscr{B}(\mathbb{R}^m)$ for all *t* and $A_{\tau} = \mathbb{R}^m$ for all $\tau > t$). The σ -field \mathscr{F}_t contains all the information about the environment that is available at date *t*.

To express precisely the fact that decisions c_t , x_t depend only on information available when the decisions are made, we simply require that c_t , x_t be measurable with respect to \mathcal{F}_t .

Formally the maximization in (1) is taken over all stochastic processes $\{c_t\}_{t=1}^{\infty}, \{x_t\}_{t=1}^{\infty}$ that satisfy (2)–(5) and such that for each $t = 1, 2, \ldots, c_t, x_t$ are measurable \mathcal{F}_t . Call such processes *admissible*.

Existence of an optimum $\{c_t\}_{t=1}^{\infty}$, $\{x_t\}_{t=1}^{\infty}$ may be established by imposing an appropriate topology \mathcal{T} on the space of admissible processes such that the objective (1) is continuous in this topology and the space of admissible processes is \mathcal{T} -compact. While it is beyond the scope of this article to discuss existence, presumably a proof can be constructed along the lines of Bewley (Ref. 19).

The notation almost makes the working of the model selfexplanatory. There are N different processes. At date t it is decided how much to consume and how much to hold in the form of capital. It is assumed that capital goods can be costlessly transformed into consumption goods on a one-for-one basis. After it is decided how much capital to hold, then it is decided how to allocate the capital across the N processes. After the allocation is decided, nature reveals the value of r_t , and $g_i(x_{it}, r_t)$ units of new production are available from process *i* at the end of period *t*. But $\delta_i x_{it}$ units of capital have evaporated at the end of *t*. Thus net new produce is $g_i(x_{it}, r_t) - \delta_i x_{it}$ from process *i*. The total produce available to be divided into consumption and capital stock at date t + 1 is given by

$$\sum_{i=1}^{N} [g_i(x_{it}, r_t) - \delta_i x_{it}] + x_t = \sum_{i=1}^{N} [g_i(x_{it}, r_t) + (1 - \delta_i) x_{it}]$$
$$\equiv \sum_{i=1}^{N} f_i(x_{it}, r_t) \equiv y_{t+1}, \tag{6}$$

where

$$f_i(x_{it}, r_t) \equiv g_i(x_{it}, r_t) + (1 - \delta_i) x_{it}$$
(7)

denotes the total amount of produce emerging from process i at the end

of period t. The produce y_{t+1} is divided into consumption and capital stock at the beginning of date t + 1, and so on.

Note that we are assuming that it is costless to install capital into each process i and that it is costless to allocate capital across processes at the beginning of each date t.

The objective of the optimizer is to maximize the expected value of the discounted sum of utilities over all consumption paths and capital allocations that satisfy (2)-(5).

To obtain sharp results we will place restrictive assumptions on this problem. We collect the basic working assumptions in one place.

Assumption 2.1. The functions $u(\cdot)$, $f_i(\cdot)$ are all concave, increasing, and twice continuously differentiable.

Assumption 2.2. The stochastic process $\{r_i\}_{i=1}^{\infty}$ is independently and identically distributed. Each $r_i:(\Omega, \mathcal{B}, \mu) \to \mathbb{R}^m$, where $(\Omega, \mathcal{B}, \mu)$ is a probability space. Here Ω is the space of elementary events, \mathcal{B} is the σ -field of measurable sets with respect to μ , and μ is a probability measure defined on subsets $B \subseteq \Omega$, $B \in \mathcal{B}$. Furthermore the range of r_i , $r_i(\Omega)$, is compact.

Assumption 2.3. For each $\{x_{i1}\}_{i=1}^{N}$, r_1 the problem (1) has a unique optimal solution (unique up to a set of realizations of $\{r_t\}$ of measure zero).

Notice that Assumption 2.3 is implied by Assumption 2.1 and strict concavity of u, $\{f_i\}_{i=1}^N$. Rather than try to find the weakest possible assumptions sufficient for uniqueness of solutions to (1), it seems simpler to reveal the role of uniqueness in what follows by simply assuming it. Furthermore since we are not interested here in the study of existence of optimal solutions, we have simply assumed existence also.

By Assumption 2.3 we see that to each output level y_t the optimum c_t , x_t , x_{it} , given y_t , may be written

$$c_t = g(y_t), \quad x_t = h(y_t), \quad x_{it} = h_i(y_t).$$
 (8)

The optimum policy functions $g(\cdot)$, $h(\cdot)$, $h_i(\cdot)$ do not depend on t because the problem given by (1)-(5) is time stationary.

Another useful optimum policy function may be obtained. Given x_t and r_t , Assumption 2.3 implies that the optimal allocation $\{x_{it}\}_{i=1}^{N}$ and the next period's optimal capital stock x_{t+1} is unique. Furthermore these may be written in the form

$$x_{it} = a_i(x_t, r_{t-1}),$$
 (9)

$$x_{t+1} = H(x_t, r_t).$$
 (10)

Equations (9) and (10) contain r_{t-1} and r_t respectively because the allocation decision is made after r_{t-1} is known but before r_t is revealed, while the capital-consumption decision is made after y_{t+1} is revealed, i.e., after r_t is known.

Equation (10) looks very much like the optimal stochastic process studied by Brock and Mirman (Ref. 2) and by Mirman and Zilcha (Refs. 20-22). For the case N = 1 the stochastic difference equation (10) converges in distribution to a unique limit distribution independent of initial conditions (Refs. 2, 3). The same result may be obtained for our *N*-process model (Ref. 7). The following result gives conditions for necessity of the transversality condition at ∞ .

Result 2.1. Assume Assumption 2.1. Also assume that units of utility may be chosen so that $u(c) \ge 0$, for all c. Furthermore assume that along optima

$$E_1\beta^{t-1}U(y_t) \to 0 \text{ as } t \to \infty.$$

If $\{c_t\}_{t=1}^{\infty}$, $\{x_t\}_{t=1}^{\infty}$, $\{x_{tit}\}_{i=1}^{N}$, t = 1, 2, ..., is optimal, then the following conditions must be satisfied:

For each i, t

$$u'(c_t) \ge \beta E_t \{ u'(c_{t+1}) f'_i(x_{it}, r_t) \},$$
(10a)

$$u'(c_t)x_{it} = \beta E_t \{ u'(c_{t+1})f'_i(x_{it}, r_t)x_{it} \},$$
(10b)

$$\lim_{t \to \infty} E_1 \{ \beta^{t-1} u'(c_t) x_t \} = 0.$$
 (10c)

Proof. See Brock (Ref. 7).

Here U(y) is the maximum of (1) given y.

2.1. The Price of Systematic Risk

Steve Ross (Ref. 8) produced a theory of capital asset pricing showing that the assumption that all systematic risk-free portfolios earn the risk-free rate of return plus the assumption that asset returns are generated by a K-factor model lead to the existence of "prices" λ_0 , λ_1 , $\lambda_2, \ldots, \lambda_K$ on mean returns and on each of the K factors. These prices satisfied the property that the expected return $E\hat{\mathcal{Z}}_i \equiv a_i$ on each asset *i* was a linear function of the standard deviation of the returns on asset *i* with respect to each factor *k*, i.e.,

$$a_i = \lambda_0 + \sum_{k=1}^{K} \lambda_k b_{ki}, \qquad i = 1, 2, \dots, N,$$
 (11)

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